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FAST TRACK COMMUNICATION

Integral formulae for the eigenvalue density of complex random matrices

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Abstract
We show that the density of complex eigenvalues for unitary invariant ensembles of complex matrices A can be written as an integral over the eigenvalues g_j of $AA^\dagger$. For the standard random matrix ensembles with matrix density of the form $\prod_j w(g_j)$, this integral can be further reduced to a twofold integral involving the Christoffel–Darboux kernel for the orthogonal polynomials associated with weight w.

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1. Introduction

Random matrices have found many interesting applications in physical sciences and beyond. Motivated by applications, random matrix theory is concerned with studying statistical patterns in the eigenvalue distribution. In this context it helps to know the joint probability density function (jpdf) of eigenvalues. For the traditional random matrix ensembles the jpdf can be calculated in a closed form [1, 2], and, consequently, one gets access to various eigenvalue statistics such as the mean eigenvalue density, eigenvalue correlation functions or spacings between eigenvalues.

Recently, random matrices with complex eigenvalues filling densely parts of the complex plane have attracted considerable interest in the mathematical and theoretical physics literature. Compared to random matrices with real eigenvalues where one can study the eigenvalue distribution by moving off the real line and avoiding singularities, matrices with complex eigenvalues present much more of a challenge as one has to deal with generating functions, such as the resolvent, inside the domain of the eigenvalue distribution. Similarly, finding the jpdf of complex eigenvalues in a closed form is generally a difficult task which has been successfully accomplished for the Gaussian matrix ensembles [3–5] and a few ensembles beyond [6, 7].

One approach that has proved to be useful for studying eigenvalue distributions in the complex plane is based on the so-called method of Hermitization [8–10], or equivalently on the use of the logarithmic potential, see [11, 12]. In this approach one reduces the complex eigenvalue problem to a one-parameter family of real eigenvalue problems and then uses the standard techniques to deal with the latter. In this communication, building on the previous work of two of us [13], we develop an alternative approach to Hermitization.

We consider the general class of invariant random matrix ensembles. Our matrices A are complex $N \times N$ and the law of their distribution is invariant under the left and right multiplication by the unitary matrices. Without loss of generality we may assume this law to be of the form

$$\mathrm{d}P(A) \propto W(g_1, \dots, g_N)\,\mathrm{d}A, \qquad 0 \leqslant a \leqslant g_j \leqslant b \leqslant +\infty \quad \text{for all} \quad j, \tag{1}$$

where the weight W is a continuous symmetric function in the eigenvalues g_j of $AA^\dagger$ and $\mathrm{d}A$ is the Cartesian volume element in the space of complex matrices, $\mathrm{d}A = \prod_{ij} \mathrm{d}\,\mathrm{Re}\,A_{ij}\mathrm{d}\,\mathrm{Im}\,A_{ij}$. The complex Ginibre ensemble [3] belongs to this class with the weight function $W = \mathrm{e}^{-\mathrm{Tr}\,AA^\dagger} = \mathrm{e}^{-\sum_j g_j}$ as well as its extension $W = \mathrm{e}^{-\mathrm{Tr}\,V(AA^\dagger)} = \mathrm{e}^{-\sum_j V(g_j)}\,\mathrm{d}A$, introduced by Feinberg and Zee [8]. Another example is provided by the Jacobi ensemble with $W = \det(AA^\dagger)^p \det(I - AA^\dagger)^q = \prod_j g_j^p (1 - g_j)^q$ which appears naturally in the context of truncations of random unitary matrices [14, 15].

It is apparent that the eigenvalues z_j of A are not completely independent of the eigenvalues g_j of $AA^\dagger$. While it is rather difficult to give a precise deterministic description of the relation between the two sets of eigenvalues, one can link the eigenvalues of $AA^\dagger$ to those of A in the stochastic setup. In the previous work [13] it was shown that the density of eigenvalues of A,

$$\rho(z) = \left\langle \frac{1}{N} \sum_{j=1}^{N} \delta(x - \mathrm{Re}\, z_j)\delta(y - \mathrm{Im}\, z_j) \right\rangle, \qquad z = x + \mathrm{i}y, \tag{2}$$

is completely determined by those of $AA^\dagger$ in the following sense. If the eigenvalues of $AA^\dagger$ are fixed, i.e. $W(g_1, \dots, g_N) = \prod_{j=1}^{N} \delta(g_j - \tilde{g}_j)$ in (1), then $\rho(x, y)$ can be expressed in terms of the $\tilde{g}_j$'s in a closed form, see theorem 2.1. This means that in the ensemble (1) with the general weight W one can obtain the density of eigenvalues by integrating over the eigenvalues of $AA^\dagger$. The corresponding integration formula is one of the two results of this communication, see theorem 2.2. The other one concerns the multiplicative weights,

$$W(g_1, \dots, g_N) = \prod_{j=1}^{N} w(g_j), \tag{3}$$

which are pertinent to the Feinberg–Zee and Jacobi ensembles. We show that in this case the density of eigenvalues of A can be obtained by integrating the Christoffel–Darboux kernel for the orthogonal polynomials associated with the weight w, see theorem 2.3.

The communication is organized as follows. In section 2, we state our main results. The details of proofs are given in section 3 and section 4 contains two examples.

2. Main results

In order to state our main results we need to recall a theorem from the previous work [13]. Let z be a complex number and $g = (g_1, \dots, g_N)$. Define functions $F_i(g)$, $i = 1, \dots, N$, by

$$F_i(g) = \frac{(g_i - |z|^2)^{N-2}}{\pi} \int_0^\infty \frac{N\,\mathrm{d}t}{(1+t)^{N+2}} \left[(N - t) + \frac{g_i}{|z|^2}(Nt - 1) \right] \prod_{j=1, j\neq i}^{N} \frac{1 + \frac{t g_j}{|z|^2}}{g_i - g_j}. \tag{4}$$

For future reference we note the following important symmetries: for each $i = 1, \ldots, N$

$$F_i(g) \text{ is invariant wrt any permutation of } g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_N \tag{5}$$

and

$$F_N(g)|_{g_i \leftrightarrow g_N} = F_i(g). \tag{6}$$

Theorem 2.1 (Wei and Fyodorov [13])**.** *Let $G = \operatorname{diag}(g_1, \ldots, g_N)$ be a fixed positive diagonal matrix, such that $0 < g_1 < \cdots < g_N < \infty$, $N \geqslant 2$ and U be a random unitary matrix drawn from the circular unitary ensemble (the unitary group $U(N)$ equipped with the Haar measure). Then the density of eigenvalues of the matrices $\sqrt{G}U$ averaged over the distribution of U is given by*

$$\rho_g(z) = \begin{cases} 0 & \text{if} \quad |z|^2 < g_1 \quad \text{or} \quad |z|^2 > g_N \\ \frac{1}{N} \sum_{i=k+1}^{N} F_i(g) & \text{if} \quad g_k < |z|^2 < g_{k+1}, \quad k = 1, \ldots, N-1. \end{cases} \tag{7}$$

Note that equation (4) has an extra factor $1/\pi$ compared to the similar one in [13]. This is due to a change in the normalization convention. In [13] the eigenvalue density was normalized to $\int \rho(z)\, \mathrm{d}(|z|^2) = 1$ while in this communication we use the normalization $\int \rho(z)\, \mathrm{d}x\, \mathrm{d}y = 1$.

Now, consider the random matrix ensembles defined in (1). We recall that the eigenvalues of A are confined to the region enclosed by two concentric circles $|z|^2 = \min_j g_j$ and $|z|^2 = \max_j g_j$ with the g_j's being the eigenvalues of $AA^\dagger$. Therefore, the eigenvalues of the matrix A drawn at random from the matrix distribution (1) all lie in the region $a \leqslant |z|^2 \leqslant b$, where a and b are as defined in (1).

In this communication we show that theorem 2.1 implies

Theorem 2.2. *The average density of the eigenvalues in the random matrix ensemble defined by (1) is given by*

$$\rho(z) = \frac{1}{Q_N} \int_a^b \mathrm{d}g_1 \cdots \int_a^b \mathrm{d}g_{N-1} \int_{|z|^2}^b \mathrm{d}g_N\, F_N(g) W(g) \prod_{1 \leqslant i < j \leqslant N} (g_j - g_i)^2, \qquad a < |z|^2 < b \tag{8}$$

with the normalization constant

$$Q_N = \int_a^b \mathrm{d}g_1 \cdots \int_a^b \mathrm{d}g_N\, W(g) \prod_{1 \leqslant i < j \leqslant N} (g_j - g_i)^2. \tag{9}$$

We also show that for the multiplicative weights (3) the N-fold integral in (8) reduces to a twofold integral as follows. Let $p_n(x)$ be the monic orthogonal polynomials associated with the weight function $w(x)$,

$$\int_a^b \mathrm{d}x\, w(x) p_n(x) p_m(x) = h_n \delta_{m,n}, \qquad p_n(x) = x^n + \cdots, \tag{10}$$

and $K_N(x, y)$ be the corresponding Christoffel–Darboux kernel,

$$K_N(x, y) = \sum_{n=0}^{N-1} \frac{p_n(x) p_n(y)}{h_n} = \frac{1}{h_{N-1}} \frac{p_N(x) p_{N-1}(y) - p_{N-1}(x) p_N(y)}{x - y}. \tag{11}$$

It is well known, see e.g. [1, 2], that K_N determines the density of the eigenvalues of $AA^\dagger$ averaged over the distribution of A,

$$\left\langle \sum_{j=1}^{N} \operatorname{Tr} \delta(x - AA^\dagger) \right\rangle_A = w(x) K_N(x, x).$$

As a simple corollary of theorem 2.2, we will show that the density of eigenvalues of A can also be expressed in terms of the kernel K_N.

Theorem 2.3. *The average density of the eigenvalues in the random matrix ensemble defined by (1), (3) is given by*

$$\rho(z) = (-1)^{N-1}|z|^2 \int_{|z|^2}^{b} \mathrm{d}x\, w(x) \int_0^{\infty} \mathrm{d}s\, \frac{(x-|z|^2)^{N-2}}{(s+|z|^2)^{N+2}} \times \left[N(x+s) - \left(\frac{xs}{|z|^2} + |z|^2 \right) \right] K_N(x,-s), \qquad a < |z|^2 < b. \tag{12}$$

Equation (12) gives the average density of complex eigenvalues in a closed form in terms of the orthogonal polynomials associated with the weight $w(x)$ on the real interval (a, b). Although it does not look simple, (12) provides an efficient way of calculating the density for small matrix dimensions N, see example 2 in section 4. We also hope that it will be helpful in the large-N limit. The complexity of the derived equation (12) can be traced down to the determinantal formula for the eigenvalue density [16]

$$\rho(z) = -\frac{1}{\pi} \lim_{\epsilon\to 0} \frac{\partial}{\partial \bar{z}} \lim_{z_b \to z} \frac{\partial}{\partial z_b} \left\langle \frac{\det[\epsilon^2 I + (zI - A)(zI-A)^\dagger]}{\det[\epsilon^2 I + (z_b I - A)(z_b I - A)^\dagger]} \right\rangle_A, \tag{13}$$

which was the starting point of the calculations in [13]. In contrast to Hermitian matrices, averages of ratios of the characteristic polynomials are rather complicated objects to study due to singularities in the denominator [17]. Interestingly, if separated, the 'bosonic' and 'fermionic' parts of the determinantal ratio in (13) are much simpler. For the ensemble of random matrices $\sqrt{G}U$ as in theorem 2.1 they were calculated in [18], see also [17]:

$$\langle \det(zI - \sqrt{G}U)(zI - \sqrt{G}U)^\dagger \rangle_U = (N+1) \int_0^{+\infty} \frac{\mathrm{d}s}{(1+s)^{N+2}} \prod_{j=1}^{N} (s|z|^2 + g_j) \tag{14}$$

and, in the limit $\epsilon \to 0$,

$$\langle \det{}^{-1}[\epsilon^2 I + (zI - \sqrt{G}U)(zI - \sqrt{G}U)^\dagger] \rangle_U = R_g(z) \ln(1/\epsilon^2) + O(1) \tag{15}$$

where $R_g(z)$ is given by the formula on the rhs of equation (7) with

$$F_i(g) := F_i^{(b)}(g) = \frac{N(N-1)}{|z|^{2(N-1)}} (|z|^2 - g_i)^{N-2} \prod_{j=1, j\neq i} \frac{1}{g_i - g_j}.$$

These formulae yield the averages over the ensemble of random matrices (1), (3) in the same way as theorem 2.1 implies theorems 2.2 and 2.3:

$$\langle \det(zI - A)(zI - A)^\dagger \rangle_A = (-1)^N (N+1) \int_0^{+\infty} \frac{p_N(-s|z|^2)}{(1+s)^{N+2}}\, \mathrm{d}s \tag{16}$$

and

$$\langle \det{}^{-1}[\epsilon^2 I + (zI - A)(zI - A)^\dagger] \rangle_A = \langle R_g(z) \rangle_A \ln(1/\epsilon^2) + O(1)$$

with

$$\langle R_g(z) \rangle_A = \frac{N-1}{|z|^{2(N-1)}} \frac{1}{h_{N-1}} \int_{|z|^2}^{b} (x - |z|^2)^{N-2} p_{N-1}(x) w(x)\, \mathrm{d}x. \tag{17}$$

Equation (16) is an immediate consequence of (14) and the Heine formula (26) and equation (17) can be obtained by repeating the calculations given in the next section with $F_i(g)$ replaced by $F_i^{(b)}(g)$. Since the functions $F_i^{(b)}(g)$ enjoy the same symmetries (5)–(6) as the $F_i(g)$'s, the derivation of (8) from (7) goes unchanged, leading to (17).

3. Proofs

Throughout this section we assume that $N \geqslant 2$. We will use the notation $\Delta(g)$ for the Vandermonde determinant:

$$\Delta(g) = \prod_{1\leqslant i<j\leqslant N} (g_i - g_j) = \det\left(g_i^{N-j}\right)_{i,j=1\ldots,N} = \det(p_{N-j}(g_i))_{i,j=1\ldots,N}. \tag{18}$$

To prove theorems 2.2 and 2.3 we need the following two lemmas.

Lemma 3.1. *The functions $F_i(g)$ add up to zero:*

$$\sum_{i=1}^{N} F_i(g) = 0. \tag{19}$$

Proof. Recall Lagrange interpolation: for any polynomial $f(x)$ of degree $N-1$ or less

$$f(x) = \sum_{i=1}^{N} f(g_i) \prod_{j=1,j\neq i}^{N} \frac{x-g_j}{g_i-g_j}.$$

This identity can be used to verify the following two summation formulae:

$$\sum_{i=1}^{N}(g_i - |z|^2)^{N-2} \prod_{j=1,j\neq i}^{N} \frac{1+\frac{t}{|z|^2}g_j}{g_i-g_j} = -\frac{t}{|z|^2}(1+t)^{N-2}$$

and

$$\sum_{i=1}^{N} g_i(g_i - |z|^2)^{N-2} \prod_{j=1,j\neq i}^{N} \frac{1+\frac{t}{|z|^2}g_j}{g_i-g_j} = (1+t)^{N-2}.$$

On rearranging terms in the square brackets in (4),

$$(N-t) + \frac{g_i}{|z|^2}(Nt-1) = N\left(1+\frac{g_i}{|z|^2}t\right) - \left(\frac{g_i}{|z|^2}+t\right), \tag{20}$$

and making use of the above summation formulae, one sees that the contribution to $\sum_i F_i(g)$ corresponding to the first term on the rhs in (20) vanishes for all t and the contribution corresponding to the second term vanishes after integrating over t. □

Lemma 3.2. *Suppose that $S(g)$ is a symmetric function in $g_1, \ldots, g_N$ and such that the integrals below converge. Then, for any $0 \leqslant \alpha < \beta \leqslant \infty$,*

$$\int_\alpha^\beta \mathrm{d}g_1 \cdots \int_\alpha^\beta \mathrm{d}g_N\, S(g)F_N(g) = 0. \tag{21}$$

Proof. By lemma 3.1, $F_N(g) = -\sum_{j=1}^{N-1} F_i(g)$. Hence, in view of (6),

$$\int_\alpha^\beta \mathrm{d}g_1 \cdots \int_\alpha^\beta \mathrm{d}g_N\, S(g)F_N(g) = -(N-1)\int_\alpha^\beta \mathrm{d}g_1 \cdots \int_\alpha^\beta S(g)F_N(g),$$

and (21) follows. □

Proof of theorem 2.2. Consider the random matrix ensemble defined by (1). The density of eigenvalues $\rho(z)$ (2) averaged over this ensemble can be related to that of theorem 2.1 by changing to the 'polar' coordinates in the matrix space. Indeed, recalling the singular value decomposition, one can write $A = V\sqrt{G}U$ where both U and V are unitary with

V being determined modulo multiplication by a diagonal unitary matrix, and G being the diagonal matrix of the ordered eigenvalues $g_1 < g_2 < \cdots < g_N$ of $AA^\dagger$. Ignoring matrices with repeated g_j's, the correspondence between A and (U, V, G) is one to one, with the Jacobian being proportional to $\Delta^2(g)$ [19]. The matrices A and $\sqrt{G}UV$ have the same set of eigenvalues, and, by the invariance of the Haar measure, the mean eigenvalue density of $\sqrt{G}U$ does not change when U gets multiplied by V. On changing to the polar coordinates (U, V, G) one then gets the desired relation

$$\rho(z) = \frac{1}{C_N}\int_a^b \mathrm{d}g_1 \int_{g_1}^b \mathrm{d}g_2 \cdots \int_{g_{N-1}}^b \mathrm{d}g_N\, W(g)\Delta^2(g)\,\rho_g(z)$$

where

$$C_N = \int_a^b \mathrm{d}g_1 \int_{g_1}^b \mathrm{d}g_2 \cdots \int_{g_{N-1}}^b \mathrm{d}g_N\, W(g)\Delta^2(g) \;= \frac{Q_N}{N!}$$

with Q_N as in (9).

Now apply theorem 2.1 to the integral and write explicitly

$$\rho(z) = \frac{(N-1)!}{Q_N}\left\{\int_a^{|z|^2} \mathrm{d}g_1 \int_{|z|^2}^b \mathrm{d}g_2 \int_{g_2}^b \mathrm{d}g_3 \cdots \int_{g_{N-1}}^b \mathrm{d}g_N\, W(g)\Delta^2(g)\sum_{i=2}^N F_i(g) + \cdots \right.$$
$$\left. + \int_a^{|z|^2} \mathrm{d}g_1 \int_{g_1}^{|z|^2} \mathrm{d}g_2 \cdots \int_{g_{N-2}}^{|z|^2} \mathrm{d}g_{N-1} \int_{|z|^2}^b \mathrm{d}g_N\, W(g)\Delta^2(g) F_N(g)\right\}.$$

On making use of the symmetries of the functions $F_i(g)$, see (5)–(6),

$$\rho(z) = \frac{(N-1)!}{Q_N}\sum_{i=1}^{N-1}\frac{1}{\mathrm{i}!(N-1-\mathrm{i})!}$$
$$\times \int_a^{|z|^2} \mathrm{d}g_1 \cdots \int_a^{|z|^2} \mathrm{d}g_i \int_{|z|^2}^b \mathrm{d}g_{i+1} \cdots \int_{|z|^2}^b \mathrm{d}g_N\, W(g)\Delta^2(g)\, F_N(g). \tag{22}$$

On the other hand,

$$\int_a^b \mathrm{d}g_1 \cdots \int_a^b \mathrm{d}g_{N-1} \int_{|z|^2}^b \mathrm{d}g_N\, W(g)\Delta^2(g)\, F_N(g)$$
$$= \left(\int_a^{|z|^2} + \int_{|z|^2}^b\right)\mathrm{d}g_1 \cdots \left(\int_a^{|z|^2} + \int_{|z|^2}^b\right)\mathrm{d}g_{N-1} \int_{|z|^2}^b \mathrm{d}g_N\, W(g)\Delta^2(g)\, F_N(g)$$
$$= \sum_{i=1}^{N-1}\frac{(N-1)!}{\mathrm{i}!(N-1-\mathrm{i})!}\int_a^{|z|^2} \mathrm{d}g_1 \cdots \int_a^{|z|^2} \mathrm{d}g_i \int_{|z|^2}^b \mathrm{d}g_{i+1}$$
$$\cdots \int_{|z|^2}^b \mathrm{d}g_N\, W(g)\Delta^2(g)\, F_N(g). \tag{23}$$

In the last step, we have used (21) with $\alpha = |z|^2$, $\beta = b$ and $S(g) = W(g)\Delta^2(g)$. Now one can verify equations (8)–(9) by comparing (22) and (23). □

Remark 1. By lemma 3.2, we also have

$$\rho(z) = -\frac{1}{Q_N}\int_a^b \mathrm{d}g_1 \cdots \int_a^b \mathrm{d}g_{N-1} \int_a^{|z|^2} \mathrm{d}g_N\, W(g)\Delta^2(g)\, F_N(g), \qquad a < |z|^2 < b. \tag{24}$$

This formula is sometimes more convenient in applications.

Next, assume that the weight $W(g)$ is multiplicative, as in equation (3). Then the normalization constant Q_N can be expressed in terms of the h_j's (10):

$$Q_N = N! \prod_{j=0}^{N-1} h_j.$$

This follows, see e.g. [1], from the identity

$$\int_a^b \mathrm{d}x_1\, w(x_1) \cdots \int_a^b \mathrm{d}x_N\, w(x_N) \det(p_j(x_i)) \det(p_j(x_i)) = N! \det\left(\int_a^b p_i(x) p_j(x) w(x)\, \mathrm{d}x\right)$$

on recalling the Vandermonde determinant (18). The derivation leading from (8) to (12) which is given below is standard and is based on the identity [20]

$$\frac{1}{Q_N} \int \mathrm{d}x_1\, w(x_1) \cdots \int \mathrm{d}x_N\, w(x_N)\ \Delta(x_1, \ldots, x_N, y_1, \ldots, y_M) \Delta(x_1, \ldots, x_N) \\ = \det(p_{N-1+i}(y_j))_{i,j=1,\ldots,M}, \tag{25}$$

which is an extension of the Heine formula, see, e.g. [21],

$$\frac{1}{Q_N} \int_a^b \mathrm{d}x_1\, w(x_1) \cdots \int_a^b \mathrm{d}x_N\, w(x_N) \prod_{j=1}^{N} (y - x_j) \Delta^2(x_1, \ldots, x_N) = p_N(y). \tag{26}$$

Proof of theorem 2.3. Note that

$$\Delta^2(g_1, \ldots, g_N) \prod_{i=1}^{N-1} \frac{s + g_i}{g_N - g_i} = (-1)^N \frac{\Delta(g_1, \ldots, g_N, -s) \Delta(g_1, \ldots, g_{N-1})}{g_N + s}.$$

On making the substitution $s = |z|^2/t$ in the integral representation (4) for $F_N(g)$ and then applying theorem 2.2, the eigenvalue density function takes the form

$$\rho(z) = \frac{(-1)^N N |z|^2}{\pi Q_N} \int_a^b \mathrm{d}g_1 \cdots \int_a^b \mathrm{d}g_{N-1} \int_{|z|^2}^b \mathrm{d}g_N \int_0^\infty \mathrm{d}s\, \frac{(g_N - |z|^2)^{N-2}}{(s + |z|^2)^{N+2}} \prod_{i=1}^{N} w(g_i) \\ \times \frac{\Delta(g_1, \ldots, g_N, -s) \Delta(g_1, \ldots, g_{N-1})}{g_N + s} \left[N(g_N + s) - \left(\frac{g_N s}{|z|^2} + |z|^2\right)\right]. \tag{27}$$

Define $g_{N+1} = -s$. Then, by (25), the integral over $g_1, \ldots, g_{N-1}$ yields

$$\int_a^b \mathrm{d}g_1 \cdots \int_a^b \mathrm{d}g_{N-1} \prod_{i=1}^{N-1} w(g_i) \frac{\Delta(g_1, \ldots, g_N, -s) \Delta(g_1, \ldots, g_{N-1})}{g_N + s} \\ = -(N-1)! \left(\prod_{n=0}^{N-1} h_n\right) K_N(g_N, -s),$$

and theorem 2.3 follows. □

Remark 2. By lemma 3.2, we also have an alternative form:

$$\rho(z) = (-1)^N |z|^2 \int_a^{|z|^2} \mathrm{d}x\, w(x) \int_0^\infty \mathrm{d}s\, \frac{(x - |z|^2)^{N-2}}{(s + |z|^2)^{N+2}} \\ \times \left[N(x + s) - \left(\frac{xs}{|z|^2} + |z|^2\right)\right] K_N(x, -s), \qquad a < |z|^2 < b. \tag{28}$$

4. Examples

In this section, we give two examples of the calculation of the eigenvalue density with the help of the formulae obtained in this communication.

Example 1 (Complex Laguerre-type ensemble)**.** Consider complex $N \times N$ matrices with the probability distribution

$$P(A) \propto \det(AA^\dagger)^a \,\mathrm{e}^{-\mathrm{Tr}\,AA^\dagger}\,\mathrm{d}A = \prod_j g_j^a \,\mathrm{e}^{-g_j}\,\mathrm{d}A, \qquad a \geqslant 0.$$

It is a simple modification of the complex Ginibre ensemble. This is a case when the alternative way of formulating theorem 2.3 becomes convenient. First, note that

$$\frac{gs}{|z|^2} + |z|^2 = (s + |z|^2) - (g - |z|^2) + \frac{(g - |z|^2)(s + |z|^2)}{|z|^2}.$$

The monic orthogonal polynomials associated with the weight $w(x) = x^a\,\mathrm{e}^{-x}$, $x > 0$, are the generalized Laguerre polynomials $L_n^a(x)$ scaled appropriately, $p_n(x) = (-1)^n n! L_n^a(x)$ [22], and the Christoffel–Darboux kernel (11) is

$$K_N(x, y) = \sum_{n=0}^{N-1} \frac{n!}{\Gamma(a + n + 1)} L_n^a(x) L_n^a(y).$$

Using the two integral formulae [23] below,

$$\int_0^\infty \frac{L_n^a(-s)}{(s + |z|^2)^m}\,\mathrm{d}s = \frac{1}{(m-1)!|z|^{2(m-1)}} \sum_{i=0}^{n} \binom{n+a}{n-i} (m - \mathrm{i} - 2)!|z|^{2i}, \qquad m \geqslant n + 2$$

$$\int_0^1 (1 - x)^{\mu-1}\,\mathrm{e}^{-\beta x} L_n^a(\beta x) = \frac{\Gamma(n + a + 1)}{n!\Gamma(a + 1)} \frac{1}{\mu}\, {}_2F_2(n + a + 1, 1; a + 1, \mu + 1; -\beta),$$

one can convert (28) into

$$\rho(z) = \frac{1}{\pi N}\,\mathrm{e}^{-|z|^2} \sum_{l=0}^{N-1} \frac{|z|^{2(l+a)}}{\Gamma(l + 1 + a)},$$

recovering Ginibre's expression [1, 3].

Example 2 (Complex Hermite-type ensemble)**.** In this example, we consider the ensemble of complex $N \times N$ matrices with the probability distribution

$$P(A) \propto \mathrm{e}^{-\mathrm{Tr}(AA^\dagger)^2}\,\mathrm{d}A = \mathrm{e}^{-\sum_j g_j^2}\,\mathrm{d}A. \qquad (29)$$

In this case the weight function is $w(x) = \mathrm{e}^{-x^2}$, $x > 0$. A general expression for the associated orthogonal polynomials seems to be unknown; however, applying the Gram–Schmidt procedure one can easily construct them one by one, e.g.

$$p_0(x) = 1 \qquad h_0 = \sqrt{\pi}/2$$

$$p_1(x) = x - 1/\sqrt{\pi} \qquad h_1 = \frac{\pi - 2}{4\sqrt{\pi}}$$

$$p_2(x) = x^2 + \frac{\sqrt{\pi}}{2 - \pi}x - \frac{4 - \pi}{4 - 2\pi} \qquad h_2 = \frac{\sqrt{\pi}}{4}\frac{\pi - 3}{\pi - 2}$$

$$p_3(x) = x^3 - \frac{3\pi - 8}{2\sqrt{\pi}(\pi - 3)}x^2 - \frac{3\pi - 10}{2\pi - 6}x + \frac{5\pi - 16}{4\sqrt{\pi}(\pi - 3)} \qquad h_3 = \frac{6\pi^2 - 29\pi + 32}{16\sqrt{\pi}(\pi - 3)}.$$

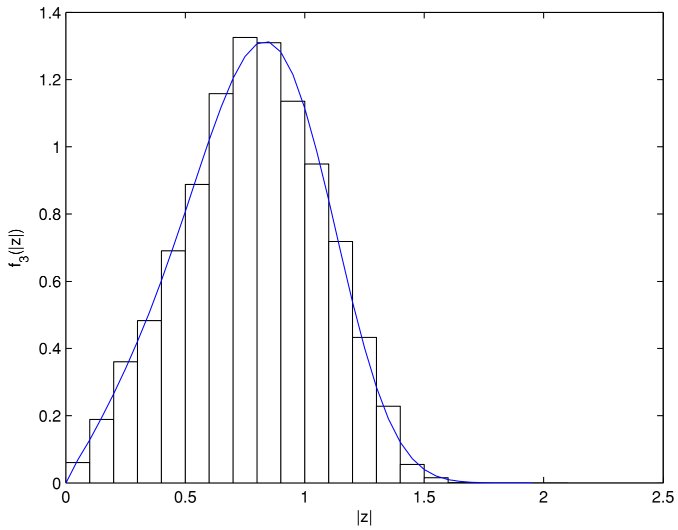

Figure 1. Histogram of the radial part of the eigenvalue distribution of 3×3 complex matrices A distributed as in (29), with sample size 50 000 and bin 0.05. Solid line represents the function $f_3(|z|)$ as derived from (30).

It is then straightforward to apply theorem 2.3 and get the first two density functions

$$\rho_2(z) = \frac{\sqrt{\pi}}{2-\pi}\left\{(\mathrm{erf}(|z|^2)-1)\left(\frac{1}{2|z|^4}-1\right)+\exp(-|z|^4)\left(\frac{1}{2|z|^4}-\frac{1}{\sqrt{\pi}|z|^2}-1\right)\right\}$$

and

$$\rho_3(z) = \frac{4}{\sqrt{\pi}}\frac{|z|^2}{\pi-3}\left\{(\mathrm{erf}(|z|^2)-1)\left(-\frac{\sqrt{\pi}}{6|z|^8}-\frac{\pi}{6|z|^6}+\frac{\pi}{4|z|^2}+\frac{\sqrt{\pi}}{3}\right)\right.$$
$$\left.+\exp(-|z|^4)\left(-\frac{\sqrt{\pi}}{6|z|^8}-\frac{\pi-2}{6|z|^6}+\frac{\sqrt{\pi}}{6|z|^4}+\frac{\pi}{6|z|^2}\right)\right\}. \tag{30}$$

Here erf is the error function: $\mathrm{erf}(x) = 2\pi^{-1/2}\int_0^x \mathrm{d}t\, \mathrm{e}^{-t^2}$.

Finally, since the density function of Hermite-type complex random matrices in example 2 is, to the best of our knowledge, not known in the literature, we compare our formula with numerical simulations. To this end, we generate 3×3 complex matrices according to the probability distribution (29). We draw a histogram of the radial part of eigenvalues, $|z|$, of the matrix A, see figure 1. To compare with the histogram, we use the appropriately normalized density function $f_3(|z|) = 2\pi|z|\rho_3(z)$, which is shown by the solid line. From figure 1, we observe a very good match between our formula (30) and the results of numerical simulations.

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